KYOTO UNIVERSITY

Statistical Learning Theory
- Regression -

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Linear Regression

Regression:
Supervised learning for predicting a real valued variable

- Regression learning is one of supervised learning problem settings with a wide range of applications
- Goal: Obtain a function $f: \mathcal{X} \to \Re$ ($\Re$: real value)
  - Usually, input domain $\mathcal{X}$ is a D-dimensional vector space
    - E.g. $x \in \mathcal{X}$ is a house  and $y \in \Re$ is its price  (housing dataset in UCI Machine Learning Repository)
- Training dataset: N pairs of an input and an output $\left\{\left(\mathbf{x}^{(1)}, y^{(1)}\right), \left(\mathbf{x}^{(2)}, y^{(2)}\right), \ldots, \left(\mathbf{x}^{(N)}, y^{(N)}\right)\right\}$
  - We use the training dataset to estimate f

Some applications of regression:
From marketing prediction to chemo-informatics

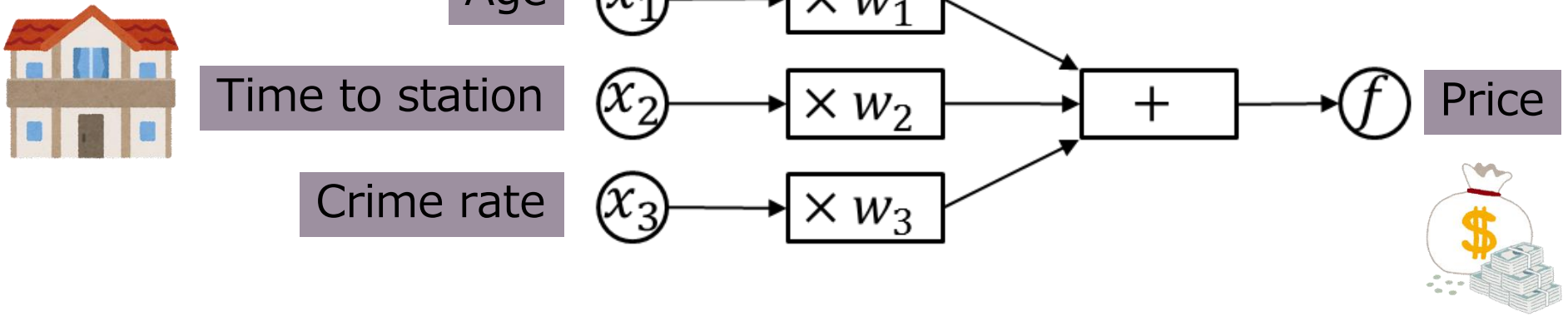
- Some applications:
  - Price prediction: Predict the price y of a product x
  - Demand prediction: Predict the demanded amount y of a product x
  - Sales prediction: Predict the sales amount y of a product x
  - Chemical activity: Predict the activity level y of a compound x
- Other applications:
  - Time series prediction: Predict the value y at the next time step given the past measurements x
  - Classification (has a discrete output domain)

A simplest model for regression: Linear regression model

- Model: How does output y depend on input $\mathbf{x}$?
- We consider the simplest choice: Liner regression model

$$y = \mathbf{w}^{\mathsf{T}}\mathbf{x} = w_1 x_1 + w_2 x_2 + \cdots + w_D x_D$$
$$\mathbf{w} = (w_1, w_2, \ldots, w_D), \mathbf{x} = (x_1, x_2, \ldots, x_D)$$

  - Example: Prediction model of the price of a house:



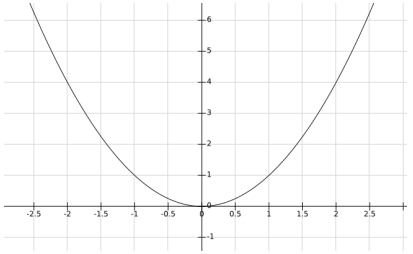
Handling discrete features: Dummy variables

- We assume input $\mathbf{x}$ is a real vector
  - In the house price prediction example, features can be age, walk time to the nearest station, crime rate in the area, …
    - They are considered as real values
- How do we handle discrete features as real values?
  - Binary features: $\{\text{Right}, \text{Left}\}$ are encoded as $\{0,1\}$
  - Called dummy variables
  - One-hot encoding: $\{\text{Kyoto}, \text{Osaka}, \text{Tokyo}\}$ are encoded with $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$

Training a linear regression model: Formulation as a squared loss minimization problem

- Objective function (to minimize):
  Disagreement measure of the model to the training dataset
  - Objective function: $L(\mathbf{w}) = \sum_{i=1}^{N} \ell\left(y^{(i)}, \mathbf{w}^\top \mathbf{x}^{(i)}\right)$
  - Loss function: $\ell\left(y^{(i)}, \mathbf{w}^\top \mathbf{x}^{(i)}\right)$ for the i-th instance
- Squared loss function is a typical choice:

$$\ell\left(y^{(i)}, \mathbf{w}^\top \mathbf{x}^{(i)}\right) = \left(y^{(i)} - \mathbf{w}^\top \mathbf{x}^{(i)}\right)^2$$



  - Absolute loss, Huber loss: more robust alternative choices
- Optimal parameter $\mathbf{w}^*$ is the one that minimizes $L(\mathbf{w})$:

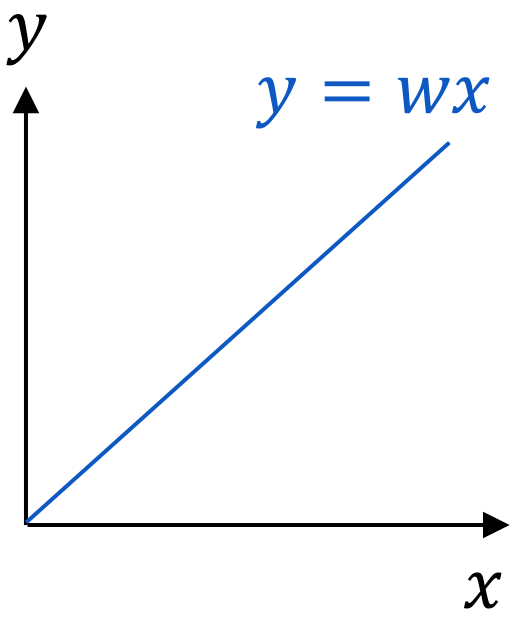
$$\mathbf{w}^* = \operatorname{argmin}_{\mathbf{w}} L(\mathbf{w})$$

Solution of linear regression problem: One dimensional case

- Let us start with a case where inputs and outputs are both one-dimensional: $y = wx$
- Objective function to minimize:

$$L(w) = \sum_{i=1}^{N} \left(y^{(i)} - wx^{(i)} \right)^2$$

- Solution: $w^* = \frac{\sum_{i=1}^{N} y^{(i)} x^{(i)}}{\sum_{i=1}^{N} x^{(i)2}} = \frac{\mathrm{Cov}(x,y)}{\mathrm{Var}(x)}$
  - Obtained by solving $\frac{\partial L(w)}{\partial w} = 0$



Solution of linear regression problem: General multi-dimensional input case

- Matrix and vector notations:
  - Design matrix $\boldsymbol{X} = \left[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(N)}\right]^\top$
  - Target vector $\mathbf{y} = \left(y^{(1)}, y^{(2)}, \ldots, y^{(N)}\right)^\top$

The Matrix Cookbook
[http://matrixcookbook.com]

Kaare Brandt Petersen
Michael Syskind Pedersen

Version: November 15, 2012

- Objective function:

$$L(\mathbf{w}) = \sum_{i=1}^{N} \left(y^{(i)} - \mathbf{w}^\top \mathbf{x}^{(i)}\right)^2 = \|\mathbf{y} - \boldsymbol{X}\mathbf{w}\|_2^2$$
$$= (\mathbf{y} - \boldsymbol{X}\mathbf{w})^\top (\mathbf{y} - \boldsymbol{X}\mathbf{w})$$

We assume the inverse exists

- Solution: $\mathbf{w}^* = \operatorname{argmin}_{\mathbf{w}} L(\mathbf{w}) = (\boldsymbol{X}^\top \boldsymbol{X})^{-1} \boldsymbol{X}^\top \mathbf{y}$
  - Equivalent to the solution of linear equations: $(\boldsymbol{X}^\top \boldsymbol{X})\mathbf{w}^* = \boldsymbol{X}^\top \mathbf{y}$

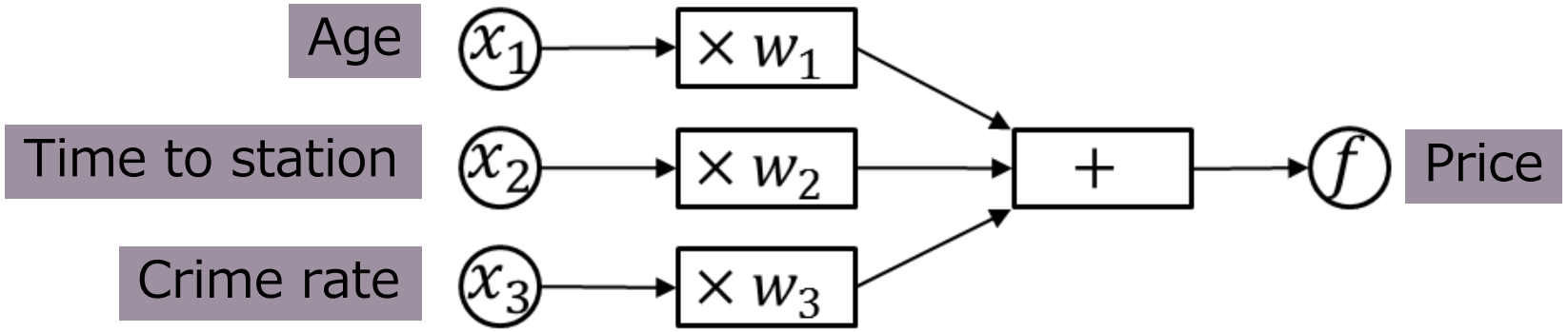
Example:
House price prediction

- Design matrix: Training data with 4 houses

$$\boldsymbol{X} = \left[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \mathbf{x}^{(3)}, \mathbf{x}^{(4)}\right]^{\mathsf{T}} = \left[\begin{pmatrix} 15 \\ 10 \\ 1.0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 0.1 \end{pmatrix}, \begin{pmatrix} 35 \\ 5 \\ 7.0 \end{pmatrix}, \begin{pmatrix} 40 \\ 70 \\ 1.0 \end{pmatrix}\right]^{\mathsf{T}}$$

- Target vector:

$$\mathbf{y} = \left(y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}\right)^{\mathsf{T}} = (140, 85, 220, 115)^{\mathsf{T}}$$



Regularization

Ridge regression:
Include penalty on the norm of $\mathbf{w}$ to avoid instability

- Existence of the solution $\mathbf{w}^* = (\boldsymbol{X}^\top \boldsymbol{X})^{-1}\boldsymbol{X}\mathbf{y}$ requires that $\boldsymbol{X}^\top \boldsymbol{X}$ is *non-singular* or *regular*, i.e. full-rank
  - This is often secured when the number of data instances N is much larger than the number of dimensions D ($N \gg D$)
- Regularization: Adding some constant $\lambda > 0$ to the diagonals of $\boldsymbol{X}^\top \boldsymbol{X}$ to make it regular (and also for numerical stability)
  - Modified solution: $\mathbf{w}^* = (\boldsymbol{X}^\top \boldsymbol{X} + \lambda \boldsymbol{I})^{-1}\boldsymbol{X}^\top \mathbf{y}$
- Back to its objective function, the new solution corresponds to

$$L(\mathbf{w}) = \|\mathbf{y} - \boldsymbol{X}\mathbf{w}\|_2^2 + \lambda\|\mathbf{w}\|_2^2$$

  - $\lambda\|\mathbf{w}\|_2^2$ is called a (L2-)regularization term

(L2-)regularization term

Generalization:

Our goal is to find a model performs well for future data

- When the number of data instances N is less than the number of dimensions D, the solution is not uniquely determined
  - Infinite number of solutions exist for $(\boldsymbol{X}^\top \boldsymbol{X})\mathbf{w}^* = \boldsymbol{X}^\top \mathbf{y}$
- All solutions equally fit to the training data

  (= minimize the loss function $L(\mathbf{w}) = \sum_{i=1}^{N} \left(y^{(i)} - \mathbf{w}^\top \mathbf{x}^{(i)}\right)^2$)
  - Some perform well, some perform badly
- Generalization: our ultimate goal is to make correct predictions for *future data*, not for the past (training) data
- Question: Which is the "best" model among them?

Occam's razor principle: Prefers simpler models

- How should we find the best model?
- Occam's razor principle: "Take the simplest model"
  - We will discuss why the simple model is good later in the "statistical learning theory"

William of Ockham

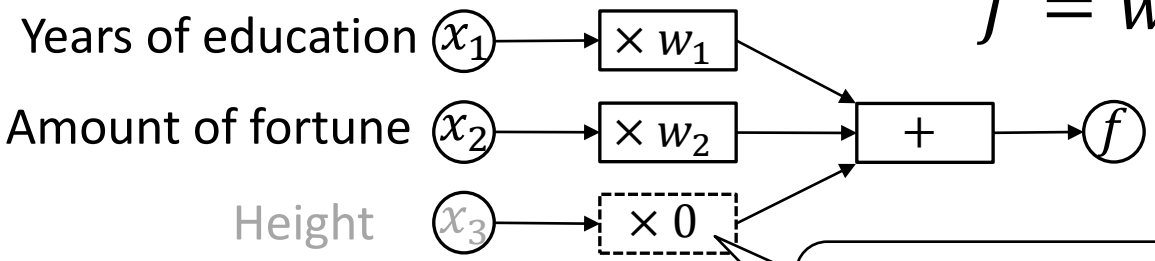
- Overfitting: "Larger" models tend to fit too much to the training data, which degrades predictive performance on future data
- What is the measure of simplicity?
  For example,
  number of features = *the number of non-zero elements in* $\mathbf{w}$
  is used as a complexity measure of a model

Occam's razor principle:
Prefers models with smaller number of variables

- Occam's razor principle prefers

Two variables

$$f = w_1 x_1 + w_2 x_2 + w_3 x_3$$

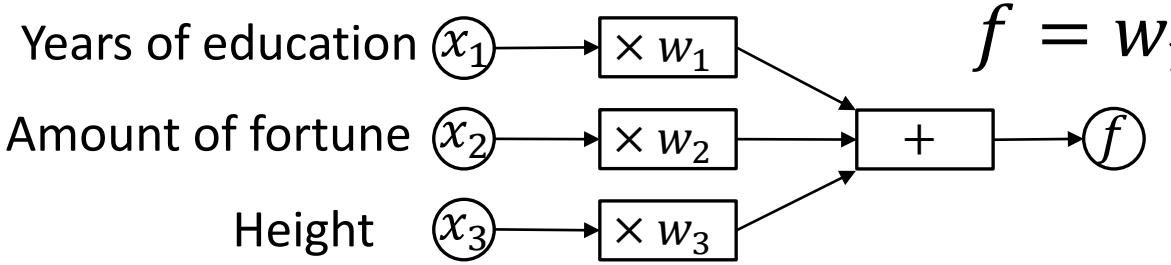


Zero weight $w_3 = 0$ is equivalent to absence of the corresponding variable x_3

to

Three variables

$$f = w_1 x_1 + w_2 x_2 + w_3 x_3$$



0-norm regularization:
Reduces the number of non-zero elements in $\mathbf{w}$

- Number of non-zero elements in $\mathbf{w}$ = "0-norm of $\mathbf{w}$"
- Use 0-*norm constraint*:

$$\text{minimize}_{\mathbf{w}} \|\mathbf{y} - \boldsymbol{X}\mathbf{w}\|_2^2 \text{ s.t. } \|\mathbf{w}\|_0 \leq \eta$$

Number of features used in the model

or equivalently, 0-*norm penalty*:

$$\text{minimize}_{\mathbf{w}} \|\mathbf{y} - \boldsymbol{X}\mathbf{w}\|_2^2 + \lambda\|\mathbf{w}\|_0$$

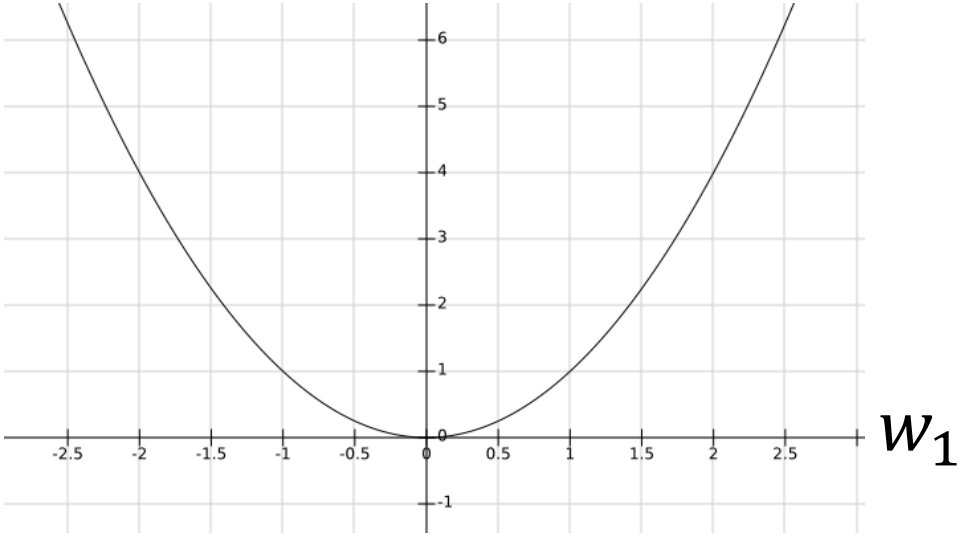
$\lambda > 0$: Regularization constant

  - There is some one-to-one correspondence between η and λ

- However, they are non-convex optimization problems …
  - 0-norm is a *non-convex* function
  - Hard to find the optimal solution

Ridge regression :
2-norm regularization as a convex surrogate for 0-nor

- Instead of the zero-norm $\|\mathbf{w}\|_0$, we use 2-norm $\|\mathbf{w}\|_2^2$



- Ridge regression: $L(\mathbf{w}) = \|\mathbf{y} - \boldsymbol{X}\mathbf{w}\|_2^2 + \lambda\|\mathbf{w}\|_2^2$

Convex ☺

  - Can be seen as a relaxed version 🤔 of

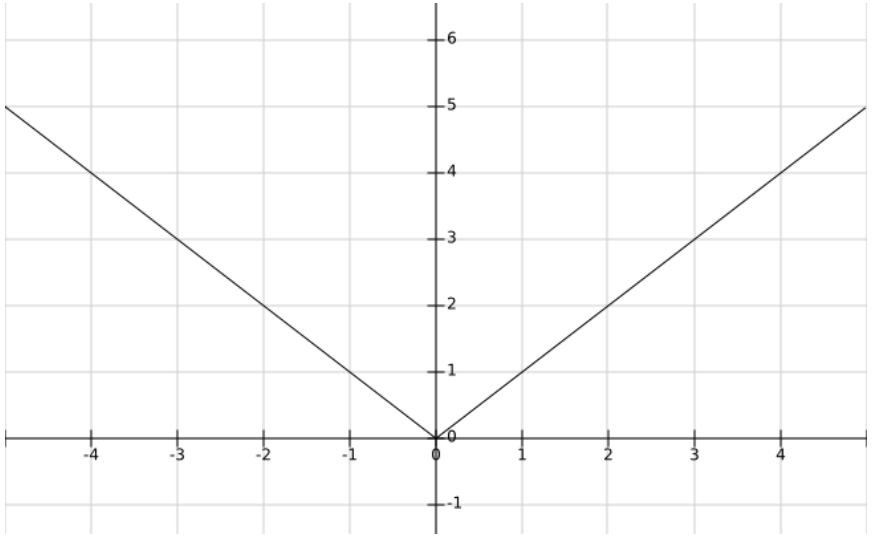
$$L(\mathbf{w}) = \|\mathbf{y} - \boldsymbol{X}\mathbf{w}\|_2^2 + \lambda\|\mathbf{w}\|_0$$

Non-convex ☹

  - The closed form solution: $\mathbf{w}^* = (\boldsymbol{X}^\top\boldsymbol{X} + \lambda\boldsymbol{I})^{-1}\boldsymbol{X}^\top\mathbf{y}$

Lasso :
1-norm regularization further induces sparsity

- Instead, we can use 1-norm $\|\mathbf{w}\|_1 = |w_1| + |w_2| + \cdots + |w_D|$



w_1

- Lasso: $L(\mathbf{w}) = \|\mathbf{y} - \boldsymbol{X}\mathbf{w}\|_2^2 + \lambda\|\mathbf{w}\|_1$
  - Convex optimization, but no closed form solution
- Sparsity inducing norm: 1-norm induces sparse $\mathbf{w}^*$

Statistical Interpretation

Interpretation as statistical inference :
Regression as maximum likelihood estimation

- So far we have formulated the regression problem in *loss minimization framework*
  - Function (prediction model) $f: \mathcal{X} \to \Re$ is deterministic
    - Linear regression model: $y = \mathbf{w}^\top \mathbf{x}$
  - Least squares: Minimization of the sum of squared losses
    - minimize $\sum_{i=1}^{N} \left(y^{(i)} - \mathbf{w}^\top \mathbf{x}^{(i)} \right)^2$
- We have not considered any statistical inference
- Actually, we can interpret the previous formulation in a statistical inference framework, namely, *maximum likelihood estimation* of a probabilistic model

Maximum likelihood estimation (MLE): Find the parameter that best reproduces training data

- We consider a probabilistic model f
  - as a conditional distribution $f_{\mathbf{w}}(y|\mathbf{x})$

Conditional probability

- We believe a good model reproduces the data well
  - "Reproduce the data well" = give high probabilities to the data
- Maximum likelihood estimation (MLE):
  - Find $\mathbf{w}$ that maximizes the likelihood function:

$$L(\mathbf{w}) = \prod_{i=1}^{N} f_{\mathbf{w}}(y^{(i)}|\mathbf{x}^{(i)})$$

    - Likelihood function: Probability that the training data is reproduced by the model
    - We assume i.i.d. (which will be explained next)

Important assumption on data generation: Identically and independently distributed

- We assume data are *identically and independently distributed*:
  - Data instances are generated from the same data generation mechanism (i.e. probability distribution)
  - Data instances are independent of each other:
    - $L(\mathbf{w}) = \prod_{i=1}^{N} f_{\mathbf{w}}(y^{(i)}|\mathbf{x}^{(i)})$
- It is often convenient to use *log* likelihood instead:

$$L(\mathbf{w}) = \sum_{i=1}^{N} \log f_{\mathbf{w}}(y^{(i)}|\mathbf{x}^{(i)})$$

- We also assume training data and future data (test data) come from the same distribution

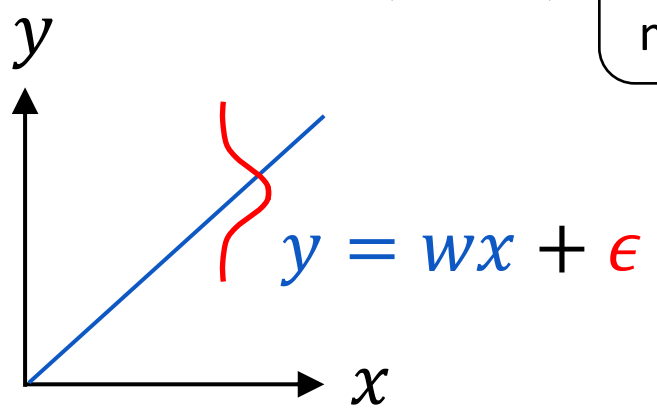
A probabilistic version of the linear regression model: Gaussian linear model

- A probabilistic version of the linear regression model $y = \mathbf{w}^\top \mathbf{x}$
- $y \sim \mathcal{N}(\mathbf{w}^\top \mathbf{x}, \sigma^2)$: a conditional distribution of y given $\mathbf{x}$
  - Gaussian distribution with mean $\mathbf{w}^\top \mathbf{x}$ and variance σ^2

$$f_{\mathbf{w}}(y|\mathbf{x}) = \mathcal{N}(\mathbf{w}^\top \mathbf{x}, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y - \mathbf{w}^\top \mathbf{x})^2}{2\sigma^2}\right)$$

  - In other words, $y = \mathbf{w}^\top \mathbf{x} + \epsilon$, where $\epsilon \sim \mathcal{N}(0, \sigma^2)$

Linear regression model



Relation between least squares and MLE: Maximum likelihood is equivalent to least squares

- Log-likelihood function (to *maximize*):

$$L(\mathbf{w}) = \sum_{i=1}^{N} \log f_{\mathbf{w}}(y^{(i)}|\mathbf{x}^{(i)})$$

$$= \sum_{i=1}^{N} \log \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\left(y^{(i)} - \mathbf{w}^\top \mathbf{x}^{(i)}\right)^2}{2\sigma^2}\right)$$

$$= -\frac{1}{2\sigma^2} \sum_{i=1}^{N} \left(y^{(i)} - \mathbf{w}^\top \mathbf{x}^{(i)}\right)^2 + \text{const.}$$

σ is treated as a constant

- Maximization of $L(\mathbf{w})$ is equivalent to *minimization* of the squared loss $\sum_{i=1}^{N} \left(y^{(i)} - \mathbf{w}^\top \mathbf{x}^{(i)}\right)^2$

Summary: Least squares regression is maximum likelihood estimation

- (Least squares) linear regression is equivalent to the maximum likelihood estimation of the Gaussian linear model
- Now, how about the ridge regression (L2-regularized linear regression)?
  - Is there any statistical inference framework corresponding to regularization?

Bayesian Statistical Interpretation

Bayesian interpretation of regression: Ridge regression as MAP estimation

- We consider another statistical interpretation of linear regression in terms of Bayesian statistics
  - Which justifies ridge (=L2-regularized) regression
- Ridge regression as MAP estimation
  - Maximum A Posteriori (MAP) estimation
  - "Posterior distribution" of parameters

Least square regression $\Longleftrightarrow$ Maximum likelihood estimation

Ridge regression $\Longleftrightarrow$ MAP estimation

Bayesian modeling: Posterior distribution instead of likelihood

- In maximum likelihood estimation (MLE), we obtain $\mathbf{w}$ that maximizes data *likelihood*:

$$P(\mathbf{y} \mid \boldsymbol{X}, \mathbf{w}) = \prod_{i=1}^{N} f_{\mathbf{w}}(y^{(i)} | \mathbf{x}^{(i)})$$

$$\text{or } \log P(\mathbf{y} \mid \boldsymbol{X}, \mathbf{w}) = \sum_{i=1}^{N} \log f_{\mathbf{w}}(y^{(i)} | \mathbf{x}^{(i)})$$

  - The probability of the data reproduced by the parameter: $P(\text{Data } \mathbf{y} \mid \text{Parameters } \mathbf{w})$

- In Bayesian modeling, we consider the *posterior distribution* $P(\text{ Parameters } \mathbf{w} \mid \text{ Data } \mathbf{y})$
  - Distribution over model parameters given data
    - In other words, model uncertainty given data

Posterior distribution:
Log posterior = log likelihood + log prior

- Posterior distribution:

$$P(\text{Parameters} \mid \text{Data}) = \frac{P(\text{Data} \mid \text{Parameters})P(\text{Parameters})}{P(\text{Data})}$$

(Bayes' formula)

- Log posterior:

$$\log P(\text{Parameters} \mid \text{Data})$$
$$= \log P(\text{Data} \mid \text{Parameters}) + \log P(\text{Parameters})$$
$$- \log P(\text{Data})$$

Likelihood

"Prior"

  - $P(\text{Parameters})$ represents our prior knowledge
  - $P(\text{Data})$ is a constant term and often neglected

Maximum a posteriori (MAP) estimation: Find parameter that maximizes the posterior

- Maximum a posteriori (MAP) estimation finds the parameter that maximizes the (log) posterior:
  $\text{Parameters}^* = \text{argmax}_{\text{Parameters}} \log P(\text{Parameters} \mid \text{Data})$
  - "Given the data, adopt the most certain parameters"
- Maximization of the log posterior:
  $\log P(\text{Parameters} \mid \text{Data})$
  $= \log P(\text{Data} \mid \text{Parameters}) + \log P(\text{Parameters}) + \text{const.}$
  - MLE considers only $\log P(\text{Data} \mid \text{Parameters})$
  - MAP has an additional term (log prior) : $\log P(\text{Parameters})$

Ridge regression as MAP estimation:
MAP with Gaussian linear model + Gaussian prior

- Log posterior: $\log P(\text{Parameters} \mid \text{Data}) = \underbrace{\log P(\text{Data} \mid \text{Parameters})}_{\text{Log likelihood}} + \underbrace{\log P(\text{Parameters})}_{\text{Log prior}} + \text{const.}$
  - Log-likelihood: $\sum_{i=1}^{N} \log \frac{1}{\sqrt{2\pi}\sigma'} \exp\left(-\frac{\left(y^{(i)} - \mathbf{w}^\top \mathbf{x}^{(i)}\right)^2}{2\sigma'^2}\right)$ (Gaussian linear model)
  - Prior $P(\mathbf{w}) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\mathbf{w}^\top \mathbf{w}}{2\sigma^2}\right)$ (Gaussian prior)
- Ridge regression is equivalent to MAP estimation:

$$\mathbf{w}^* = \operatorname{argmin}_{\mathbf{w}} \frac{1}{2\sigma'^2} \sum_{i=1}^{N} \left(y^{(i)} - \mathbf{w}^\top \mathbf{x}^{(i)}\right)^2 + \frac{1}{2\sigma^2} \|\mathbf{w}\|_2^2$$

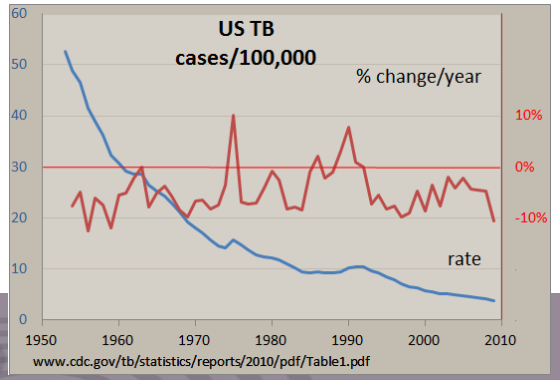
Some More Applications

Time series prediction: Auto regressive (AR) model

- Time series data: A sequence of real valued data $x_1, x_2, \ldots, x_t, \ldots \in \Re$ associated with time stamps $t = 1,2, \ldots$
- Time series prediction: Given $x_1, x_2, \ldots, x_{t-1}$, predict x_t
- Auto regressive (AR) model:

$$x_t = w_1 x_{t-1} + w_2 x_{t-2} + \cdots + w_D x_{t-D}$$

  - x_t is determined by the recent length-D history
- AR model is a linear regression model $y = \mathbf{w}^\top \mathbf{x}$:
  - $\mathbf{w} = (w_1, w_2, \ldots, w_D)^\top$
  - $\mathbf{x} = (x_{t-1}, x_{t-2}, \ldots, x_{t-D})^\top$



Classification as regression:
Regression is also applicable to classification

- Binary classification: $y \in \{+1, -1\}$
- Apply regression to predict $y \in \{+1, -1\}$
- Rigorously, such application is not valid
  - Since an output is only either $+1$ or -1 (no intermediate value), the Gaussian noise assumption does not hold
  - However, since solution of regression is often easier than that of classification, this application can be compromise
- Fisher discriminant: Instead of $\{+1, -1\}$, use $\left\{+\frac{1}{N^+}, -\frac{1}{N^-}\right\}$
  - $N^+(N^-)$ is the number of positive (negative) data instances

Nonlinear Regression

Nonlinear regression: Introducing nonlinearity in linear models

- So far we have considered only linear models
- How to introduce non-linearity in the models?
  1. Introduce nonlinear basis functions:
     - Transformed features: e.g. $x \rightarrow \log x$
     - Cross terms: e.g. $x_1, x_2 \rightarrow x_1 x_2$
     - Kernels: $\mathbf{x} \rightarrow \boldsymbol{\phi}(\mathbf{x})$ (some nonlinear mapping to a high-dimensional space)
  2. Intrinsically nonlinear models:
     - Regression tree / random forest
     - Neural network

Nonlinear transformation of features: Simplest way to introduce nonlinearity in linear models

- Nonlinear basis function: $x \to \log x\,, e^x, x^2, \frac{1}{x}, \ldots$
  - Sometimes used for converting the range
    - E.g. $\log: \Re^+ \to \Re$, $\exp: \Re \to \Re^+$
- Interpretations of log transformation:

	y	$\log y$
x	$y = \beta x + \alpha$ Increase of x by 1 will increase y by β	$\log y = \beta x + \alpha$ Increase of x by 1 will multiply y by $1 + \beta$
$\log x$	$y = \beta \log x + \alpha$ Doubling x will increase y by β	$\log y = \beta \log x + \alpha$ Doubling x will multiply y by $1 + \beta$

Cross terms:
Can include synergetic effects among different features

- Not only the original features $x_1, x_2, \ldots, x_D$, use their cross terms products $\{x_d x_{d'}\}_{d,d'}$
- Model has a matrix parameter $\boldsymbol{W}$:

$$y = \mathrm{Trace}\left(\begin{bmatrix} w_{1,1} & \cdots & w_{1,D} \\ \vdots & \ddots & \vdots \\ w_{D,1} & \cdots & w_{D,D} \end{bmatrix}^\top \begin{bmatrix} x_1^2 & x_1 x_2 & \cdots & x_1 x_D \\ x_2 x_1 & x_2^2 & & x_2 x_D \\ & \vdots & \ddots & \vdots \\ x_D x_1 & x_D x_2 & \cdots & x_D^2 \end{bmatrix} \right)$$

$$= \mathbf{x}^\top \boldsymbol{W}^\top \mathbf{x}$$

- $L(\boldsymbol{W}) = \sum_{i=1}^{N} \left(y^{(i)} - \mathbf{x}^{(i)\top} \boldsymbol{W}^\top \mathbf{x}^{(i)} \right)^2 + \lambda \|\boldsymbol{W}\|_{\mathrm{F}}^2$

(e.g. factorization machines)

Kernels:
Linear model in a high-dimensional feature space

- High dimensional non-linear mapping: $\mathbf{x} \rightarrow \boldsymbol{\phi}(\mathbf{x})$
  - $\boldsymbol{\phi}: \Re^D \rightarrow \Re^{\bar{D}}$ is some *nonlinear* mapping from D-dimensional space to a $\bar{D}$-dimensional space ($D \ll \bar{D}$)
- (Very high dimensional) linear model $y = \bar{\mathbf{w}}^\top \boldsymbol{\phi}(\mathbf{x})$
- Kernel regression model: $y = \sum_{i=1}^{N} \alpha^{(i)} k\left(\mathbf{x}^{(i)}, \mathbf{x}\right)$
  - Kernel function $k\left(\mathbf{x}^{(i)}, \mathbf{x}\right) = \langle \boldsymbol{\phi}\left(\mathbf{x}^{(i)}\right), \boldsymbol{\phi}(\mathbf{x}) \rangle$: inner product
  - Kernel trick: Instead of working in the $\bar{D}$-dimensional space, we use an equivalent form in an N -dimensional space
    - Foundation of kernel machines, e.g. SVM, Gaussian process, ...

Regression:
Supervised learning for predicting a real valued variable

- A supervised learning problem to make real-valued predictions
- Regression problem is often formulated as a least-square minimization problem
  - Closed form solution is given
- Regularization framework to avoid overfitting
  - Reduce the number of features: 0-norm, 2-norm (ridge regression), 1-norm (lasso)
- Statistical interpretations: maximum likelihood estimation, maximum a posteriori (MAP) estimation
- Nonlinear regression